A self-dual poset on objects counted by the Catalan numbers

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Abstract

We examine the poset P of 132-avoiding n-permutations ordered by descents. We show that this poset is the "coarsening" of the well-studied poset Q of noncrossing partitions . In other words, if $x < y$ in Q, then $f(y) < f(x)$ in P, where f is the canonical bijection from the set of noncrossing partitions onto that of 132-avoiding permutations. This enables us to prove many properties of P.

1 Introduction

There are more than 150 different objects enumerated by Catalan numbers. Two of the most carefully studied ones are noncrossing partitions and 132-avoiding permutations. A partition $\pi = (\pi_1, \pi_2, \cdots, \pi_t)$ of the set $[n] = \{1, 2, \cdots, n\}$ is called noncrossing [2] if it has no four elements $a < b < c < d$ so that $a, c \in \pi_i$ and $b, d \in \pi_j$ for some distinct i and j. A permutation of $[n]$, or, in what follows, an n-permutation, is called 132-avoiding [4] if it does not have three entries $a < b < c$ so that a is the leftmost of them and b is the rightmost of them.

Noncrossing partitions of $[n]$ have a natural and well studied partial order: the refinement order Q_n. In this order $\pi_1 < \pi_2$ if each block of π_2 is the union of some blocks of π_1. The poset Q_n is known to be a lattice, and it is graded, rank-symmetric, rank-unimodal, and k-Sperner [6]. The poset Q_n has been proved to be self-dual in two steps [2], [5].

In this paper we introduce a new partial order of 132-avoiding n-permutations which will naturally translate into one of noncrossing partitions. In this poset, for two 132-avoiding n-permutations x and y, we define $x < y$ if the descent set of x is contained in that of y. (We will provide a natural equivalent, definition, too.) We will see that this new partial order P_n is a coarsening of the dual of Q_n. In other words, if for two noncrossing partitions π_1 and π_2 we have $\pi_1 < \pi_2$ in Q_n, then we also

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have $f(\pi_2) < f(\pi_1)$ in P_n, where f is a natural bijection from the set of noncrossing partitions onto that of 132-avoiding permutations. This will enable us to prove that P_n has the same rank-generating function as Q_n, and so P_n is rank-unimodal, rank-symmetric and k-Sperner. Furthermore, we will also prove that P_n is self-dual in a somewhat more direct way than it is proved for Q_n.

2 Our main results

2.1 A bijection and its properties

It is not difficult to find a bijection from the set of noncrossing partitions of $[n]$ onto that of 132-avoiding n-permutations. However, we will exhibit such a bijection here and analyze its structure as it will be our major tool in proving our theorems. To avoid confusion, integers belonging to a partition will be called *elements*, while integers belonging to a permutation will be called *entries*. An n-permutation $x = x_1 x_2 \cdots x_n$ will always be written in the one-line notation, with x_i denoting its ith entry.

Let π be a noncrossing partition of $[n]$. We construct the 132-avoiding permutation $p = f(\pi)$ corresponding to it. Let k be the largest element of π which is in the same block of π as 1. Put the entry n of p to the kth position, so $p_k = n$. As p is to be 132-avoiding, this implies that entries larger than $n-k$ are on the left of n and entries less than or equal to $n-k$ are on the right of n in q.

Then we continue this procedure recursively. As π is noncrossing, blocks which contain elements larger than k cannot contain elements smaller than k. Therefore, the restriction of π to $\{k+1, k+2, \cdots, n\}$ is a noncrossing partition, and it corresponds to the 132-avoiding permutation of $\{1, 2, \cdots n-k\}$ which is on the left of n in π by this same recursive procedure.

We still need to say what to do with blocks of π containing elements smaller than or equal to k. Delete k, and apply this same procedure for the resulting noncrossing partition on $k-1$ elements. This way we obtain a 132-avoiding permutation of $k-1$ elements, and this is what we needed for the part of p on the left of n, that is, for $\{n-k+1, n-k+2, \cdots, n-1\}$.

So in other words, if π_1 is the restriction of π into $[k-1]$ and π_2 is the restriction of π into $\{k+1, k+2, \cdots, n\}$, then $f(\pi)$ is the concatenation of $f(\pi_1)$, n and $f(\pi_2)$, where $f(\pi_1)$ permutes the set $\{n-k+1, n-k+2, \cdots, n-1\}$ and $f(\pi_2)$ permutes the set $[n-k]$.

To see that this is a bijection note that we can recover the largest element of the block containing the entry 1 from the position of n in p and then proceed recursively.

Example 1 If $\pi = (\{1,4,6\}, \{2,3\}, \{5\}, \{7,8\})$, then $f(\pi) = 64573812$.

Example 2 If $p = (\{1, 2, \cdots, n\})$, then $f(p) = 12 \cdots n$.

Example 3 If $p = (\{1\}, \{2\}, \cdots, \{n\})$, then $f(p) = n \cdots 21$.

The following definition is widely used in the literature.

Definition 1 *Let $p = p_1 p_2 \cdots p_n$ be a permutation. We say the i is a* descent *of p if $p_i > p_{i+1}$. The set of all descents of p is called the* descent set *of p and is denoted $D(p)$.*

Now we are in a position to define the poset P_n of 132-avoiding permutations we want to study.

Definition 2 *Let x and y be two 132-avoiding n-permutations. We say that $x <_P y$ (or $x < y$ in P_n) if $D(x) \subset D(y)$.*

Clearly, P_n is a poset as inclusion is transitive. It is easy to see that in 132-avoiding permutations, $i \leq 1$ is a descent if and only if p_{i+1} is smaller than every entry on its left, (such an element is called a left-to-right minimum). So $x <_P y$ if and only if the set of positions in which x has a left-to-right minimum is a proper subset of that of those positions in which y has a left-to-right minimum. The Hasse diagram of P_4 is shown on the Figure below.

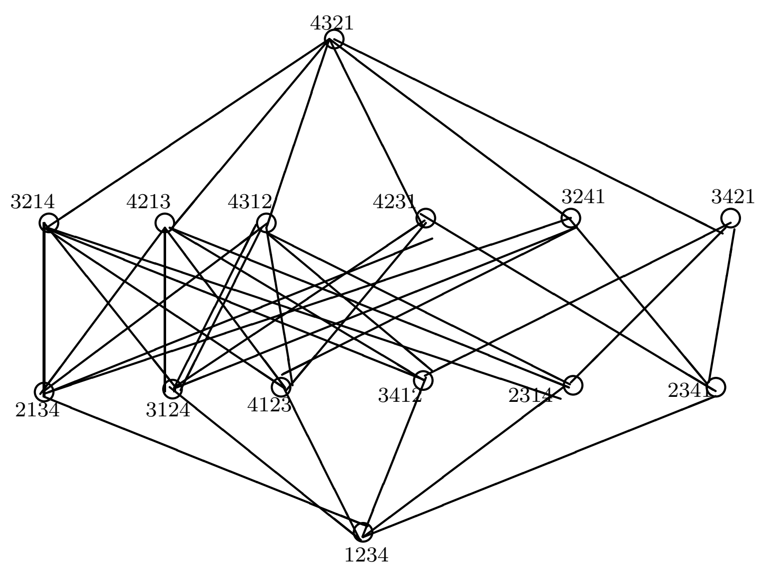

Figure 1: The Hasse diagram of P_4.

The following proposition describes the relation between the blocks of x and the descent set of $f(x)$.

Proposition 1 *The bijection f has the following property: $i \in D(f(x))$ if and only if $i+1$ is the smallest element of its block.*

Proof: By induction on n. For $n = 1$ and $n = 2$ the statement is true. Now suppose we know the statement for all positive integers smaller than n. Then we distinguish two cases:

1. If 1 and n are in the same block of x, then the construction of $f(x)$ simply starts by putting the entry n to the last slot of $f(x)$, then deleting the element n from x. Neither of these steps alters the set of minimal elements of blocks or that of descents in any way. Therefore, the algorithm is reduced to one of size $n - 1$, and the proof follows by induction.

2. If the largest element k of the block containing 1 is smaller than n, then as we have seen above, f constructs the images of x_1 and x_2 which will be separated by the entry n. Therefore, by the induction hypothesis, the descents of $f(x)$ are given by the minimal elements of the blocks of x_1 and x_2, and these are exactly the blocks of x. There will also be a descent at k (as the entry n goes to the kth slot), and that is in accordance with our statement as $k + 1$ is certainly the smallest element of its block.

$\diamond$

We point out that this implies that P_n is equivalent to a poset of noncrossing partitions in which $\pi_1 < \pi_2$ if the set of elements which are minimal in their block in π_1 is contained in that of elements which are minimal in their block in π_2.

2.2 Properties of P_n

Now we can prove the main result of this paper.

Theorem 1 *The poset P_n is coarser than the dual of the poset Q_n of noncrossing partitions ordered by refinement. That is, if $x < y$ in Q_n, then $f(y) < f(x)$ in P_n.*

Proof: If $x < y$, then each block of x is a subset of a block of y. Therefore, if z is the minimal element of a block B of y, then it is also the minimal element of the block E of x containing it as $E \subseteq B$. Therefore, the set of elements which are minimal in their respective blocks in x contains that of elements which are minimal in their respective blocks in y. By Proposition 1 this implies $D(f(y)) \subset D(f(x))$. $\diamond$

Now we apply this result to prove some properties of P_n. For definitions, see [7].

Theorem 2 *The rank generating function of P_n is equal to that of Q_n. In particular, P_n is rank-symmetric, rank-unimodal and k-Sperner.*

Proof: By proposition 1, the number of 132-avoiding permutations having k descents equals that of noncrossing partitions having k blocks, and this is known to be the (n, k) Narayana-number $\frac{1}{n} \cdot \binom{n}{k}\binom{n}{k-1}$.

Therefore P_n is graded, rank-symmetric and rank-unimodal, and its rank generating function is the same as that of Q_n, as Q_n too is graded by the number of blocks (and is self-dual). As P_n is coarser than Q_n, any antichain of P_n is an antichain of Q_n, and the k-Sperner property follows. $\diamond$

We need more analysis to prove that P_n is self-dual, that is, that P_n is invariant to "being turned upside down". Denote $Perm_n(S)$ the number of 132-avoiding n-permutations with descent set S. The following lemma is the base of our proof of self-duality. For $S \subseteq [n-1]$, we define $\alpha(S)$ to be the "reverse complement" of S, that is, $i \in \alpha(S) \Longleftrightarrow n-i \notin S$.

Lemma 1 *For any $S \subseteq [n-1]$, we have $Perm_n(S) = Perm_n(\alpha(S))$.*

Proof: By induction on n. For $n = 1, 2, 3$ the statement is true. Now suppose we know it for all positive integers smaller than n. Denote t the smallest element of S.

1. Suppose that $t > 1$. This means that $x_1 < x_2 < \cdots < x_t$, and that $x_1, x_2, \cdots, x_t$ are *consecutive integers*. Indeed, if there were a gap among them, that is, there were an integer y so that $y \neq x_i$ for $1 \leq i \leq t$, while $x_1 < y < x_t$, then $x_1\, x_t\, y$ would be a 132-pattern. So once we know x_1, we have noly one choice for $x_2, x_3, \cdots, x_t$. This implies

   $$Perm_n(S) = Perm_{n-(t-1)}(S - (t-1)), \tag{1}$$

   where $S - (t-1)$ is the set obtained from S by subtracting $t-1$ from each of its elements.

   On the other hand, we have $n-t+1, n-t+2, \cdots, n-1 \in \alpha(S)$, meaning that $x_{n-t+1} > x_{n-t+2} > \cdots > x_n$, and also, we must have $(x_{n-t+2}, \cdots, x_n) = (t-1, t-2, \cdots 1)$, otherwise a 132-pattern is formed. Therefore,

   $$Perm_n(\alpha(S)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1)) \tag{2}$$

   where $\alpha(S)|n-(t-1)$ is simply $\alpha(S)$ without its last $t-1$ elements. Clearly, $Perm_{n-(t-1)}(S - (t-1)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1))$ by the induction hypothesis, so equations (1) and (2) imply $Perm_n(S) = Perm_n(\alpha(S))$.

2. If $t = 1$, but $S \neq [n-1]$, then let u be the smallest index which is not in S. Then again, x_u must be the smallest positive integer a which is larger than x_{u-1} and is not equal to some x_i, $i \leq u-1$, otherwise $x_{u-1}\, x_u\, a$ would be a 132-pattern. So again, we have only one choice for x_u. On the other hand, the largest index in $\alpha(S)$ will be $n-(u-1)$. Then as above, we will only have once choice for x_{n-u}. Now we can delete u from S and $n-u$ from $\alpha(S)$ and proceed by the induction hypothesis as in the previous case.

3. Finally, if $S = [n-1]$, then the statement is trivially true as $Perm_n(S) = Perm_n(\alpha(S)) = 1$.

So we have seen that $Perm_n(S) = Perm_n(\alpha(S))$ in all cases. $\diamond$

Now we are in position to prove our next theorem.

Theorem 3 *The poset P_n is self-dual.*

Proof: It is clear that in P_n permutations with the same descent set will cover the same elements and they will be covered by the same elements. Therefore, such permutations form orbits of $Aut(P_n)$ and they can be permuted among each other arbitrarily by elements of $Aut(P_n)$. One can think of P_n as a Boolean algebra B_{n-1} in which some elements have several copies. One natural anti-automorphism of a Boolean-algebra is "reverse complement", that is, for $S \subseteq [n-1]$, $i \in \alpha(S) \iff n-i \notin S$. To show that P_n is self-dual, it is therefore sufficient to show that the corresponding elements appear with the same multiplicities in P_n. So in other words we must show that there are as many 132-avoiding permutations with descent set S as there are with descent set $\alpha(S)$. And that has been proved in the Lemma. $\diamond$

2.3 Further directions

It is natural to ask for what related combinatorial objects we could define such a natural partial order which would turn out to be self-dual and possibly, have some other nice properties. *Two-stack sortable permutations* [8] are an obvious candidate. It is known [1] that there are as many of them with k descents as with $n-1-k$ descents, however, the poset obtained by the descent ordering is not self-dual, even for $n = 4$, so another ordering is needed. Another candidate could be the poset of the recently introduced noncrossing partitions for classical reflection groups [3], some of which are self-dual in the traditional refinement order.

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